Problem 1 (15 points)

The series is a convergent geometric series so that

$$|e^c| = e^c < 1 \text{ thus we must have } c < 0$$

$$\sum_{n=0}^{+\infty} e^{nc} = \frac{1}{1-e^c} = D \Rightarrow 1-e^c = \frac{1}{D} \Rightarrow 1-\frac{1}{D} = e^c$$

$$c = \ln\left(1-\frac{1}{D}\right) < 0 \text{ as it should because } D > 1.$$

Problem 2

$$a)\lim_{n\to\infty}\left\{\left(\sin\frac{1}{n^{3.5}}\right)/\left(\frac{1}{n^{3.5}}\right)\right\}=\lim_{n\to\infty}\frac{\frac{d}{dn}\left(\sin\frac{1}{n^{3.5}}\right)}{\frac{d}{dn}\left(\frac{1}{n^{3.5}}\right)}=\lim_{n\to\infty}\frac{\cos\frac{1}{n^{3.5}}\frac{d}{dn}\left(\frac{1}{n^{3.5}}\right)}{\frac{d}{dn}\left(\frac{1}{n^{3.5}}\right)}$$

10 points

$$=\lim_{n\to\infty}\frac{\cos\frac{1}{n^{3.5}}\left(-3.5n^{-4.5}\right)}{\left(-3.5n^{-4.5}\right)}=\lim_{n\to\infty}\cos\frac{1}{n^{3.5}}=1>0$$

$b)\ \sum_{n=1}^{+\infty} n^{3.5}\sin\frac{1}{n^{3.5}}$ is divergent because $\lim_{n\to\infty} a_n = 1$ different than 0.

5 points

$c)\ \sum_{n=1}^{+\infty}\frac{1}{n^{3.5}}$ is convergent p-series (p = 3.5 > 1)

Taking into account the result from a) where $\lim_{n\to\infty}\left\{\left(\sin\frac{1}{n^{3.5}}\right)/\left(\frac{1}{n^{3.5}}\right)\right\}=1>0$

then the Limit comparison tests ensures also that

$\sum_{n=1}^{+\infty}\sin\frac{1}{n^{3.5}}$ is convergent

10 points

Problem 3

If $a_n = \dfrac{(5x-4)^n}{n^3}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(5x-4)^{n+1}}{(n+1)^3}\cdot\frac{n^3}{(5x-4)^n}\right| = \lim_{n\to\infty}|5x-4|\left(\frac{n}{n+1}\right)^3 = \lim_{n\to\infty}|5x-4|\left(\frac{1}{1+1/n}\right)^3$$

$$= |5x-4|\cdot 1 = |5x-4|$$

By the Ratio Test, $\sum_{n=1}^{\infty}\frac{(5x-4)^n}{n^3}$ converges when $|5x-4|<1 \Leftrightarrow \left|x-\frac{4}{5}\right|<\frac{1}{5} \Leftrightarrow -\frac{1}{5}<x-\frac{4}{5}<\frac{1}{5} \Leftrightarrow$

$\frac{3}{5}<x<1$, so $R=\frac{1}{5}$. When $x=1$, the series $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent p-series ($p=3>1$). When $x=\frac{3}{5}$, the series

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$ is absolutely convergent as a p-series with p=3.

For both x=1 and x=3/5 the series is absolutely convergent as a p-series with p=3.

<u>In conclusion</u>:

a) The series is absolutely convergent for $3/5 \le x \le 1$ 15 points

b) It is divergent for $x>1$ and $x<3/5$ 5 points

Problem-4 (30 points)

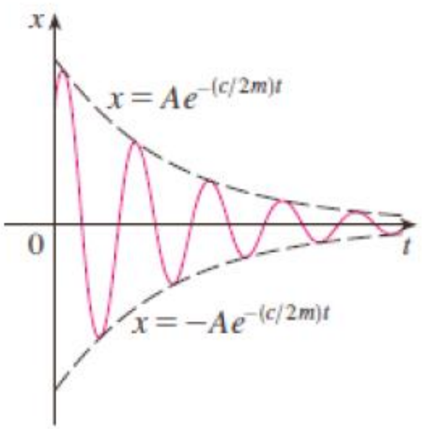
For the solution of the homogenous

■ **CASE III** $c^2 - 4mk < 0$ **(underdamping)**

Here the roots are complex:

10- points

$$\left.\begin{array}{l} r_1 \\ r_2 \end{array}\right\} = -\frac{c}{2m} \pm \omega i$$



The solution of the homogenus equation $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$ has the form (assuming $c^2 - 4mk < 0$)

$$X_{homo}(t) = C_1 e^{-t/t_0} \cos \omega' t + C_2 e^{-t/t_0} \sin \omega' t$$

$t_0 = -\frac{2m}{c}$ and $\omega' = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$ or

$$\omega' = \omega \sqrt{1 - \left(\frac{c}{2m\omega}\right)^2}$$

In general we have for the particular solution

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F_0 \cos \omega_0 t \quad (1)$$

$$x_p(t) = A(\omega_0) \cos(\omega_0 t) + B(\omega_0) \sin(\omega_0 t)$$

Substitute in (1) $\Rightarrow$

$$m(-A\omega_0^2 \cos \omega_0 t - B\omega_0^2 \sin \omega_0 t) + (-AC\omega_0 \sin \omega_0 t + BC\omega_0 \cos \omega_0 t) + k(A \cos \omega_0 t + B \sin \omega_0 t) = F_0 \cos \omega_0 t \Rightarrow$$

$$[(k - m\omega_0^2)A + cB\omega_0] \cos \omega_0 t + [(k - m\omega_0^2)B - CA\omega_0] \sin \omega_0 t = F_0 \cos \omega_0 t \quad (2)$$

$$(k - m\omega_0^2) A + CB\omega_0 = F_0 \quad (3)$$

$$(k - m\omega_0^2) B - CA\omega_0 = 0 \quad (4)$$

since $\omega = \omega_o$ (and as a result $k - m\omega^2 = 0$) we obtain after substition into the equation of motion : $c\omega \mathrm{B} = \mathrm{F}_o$ and $\mathrm{A} = 0$

$$\Rightarrow x_p(t) = \left(\frac{F_o}{c\omega}\right) \sin(\omega t)$$

20 points

Full solution:

$$x(t) = \mathrm{e}^{-(c/2m)t}[\mathrm{c}_1 \cos(\tilde{\omega} t) + \mathrm{c}_2 \sin(\tilde{\omega} t)] + \left(\frac{F_o}{c\omega}\right) \sin(\omega t)$$

$$\tilde{\omega} = \omega\sqrt{1 - (c/2m\omega)^2}$$